# D F T

## (Discrete Fourier Transform)

# F F T

## (Fast Fourier Transform)

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### Introduction

This document describes the Discrete Fourier Transform (DFT), that is, a Fourier Transform as applied to a discrete complex valued series. The mathematics will be given and source code (written in the C programming language) is provided in the appendices.

### Theory

### Continuous

For a continuous function of one variable f(t), the Fourier Transform F(f) will be defined as:

$$F(f) = \int_{-\infty}^{\infty} f(t)\ e^{-j2\pi ft}\,dt$$

and the inverse transform as

$$f(t) = \int_{-\infty}^{\infty} F(f)\ e^{j2\pi ft}\,df$$

where j is the square root of -1 and e denotes the natural exponent

$$e^{j\phi} = \cos(\phi) + j\sin(\phi).$$

### Discrete

Consider a complex series x(k) with N samples of the form

$$x_0,\ x_1,\ x_2,\ x_3\ \ldots\ x_k\ \ldots\ x_{N-1}$$

where x is a complex number

$$x_i = x_{real} + j\, x_{imag}$$

Further, assume that that the series outside the range 0, N-1 is extended N-periodic, that is, $x_k = x_{k+N}$ for all k. The FT of this series will be denoted X(k), it will also have N samples. The forward transform will be defined as

$$X(n) = \frac{1}{N} \sum_{k=0}^{N-1} x(k)\; e^{-jk2\pi n/N} \quad \text{for } n=0..N-1$$

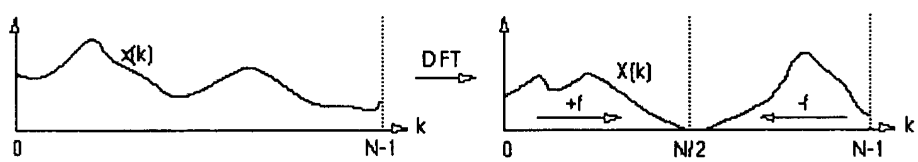
The inverse transform will be defined as

$$x(n) = \sum_{k=0}^{N-1} X(k)\; e^{jk2\pi n/N} \quad \text{for } n=0..N-1$$

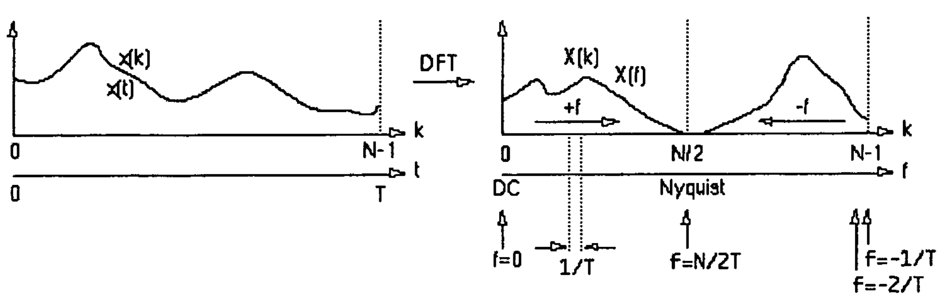
Of course although the functions here are described as complex series, real valued series can be represented by setting the imaginary part to 0. In general, the transform into the frequency domain will be a complex valued function, that is, with magnitude and phase.

$$\text{magnitude} = ||X(n)|| = ( x_{real} * x_{real} + x_{imag} * x_{imag} )^{0.5}$$
$$\text{phase} = \tan^{-1}\left(\frac{x_{imag}}{x_{real}}\right)$$

The following diagrams show the relationship between the series index and the frequency domain sample index. Note the functions here are only diagramatic, in general they are both complex valued series.




For example if the series represents a time sequence of length T then the following illustrates the values in the frequency domain.



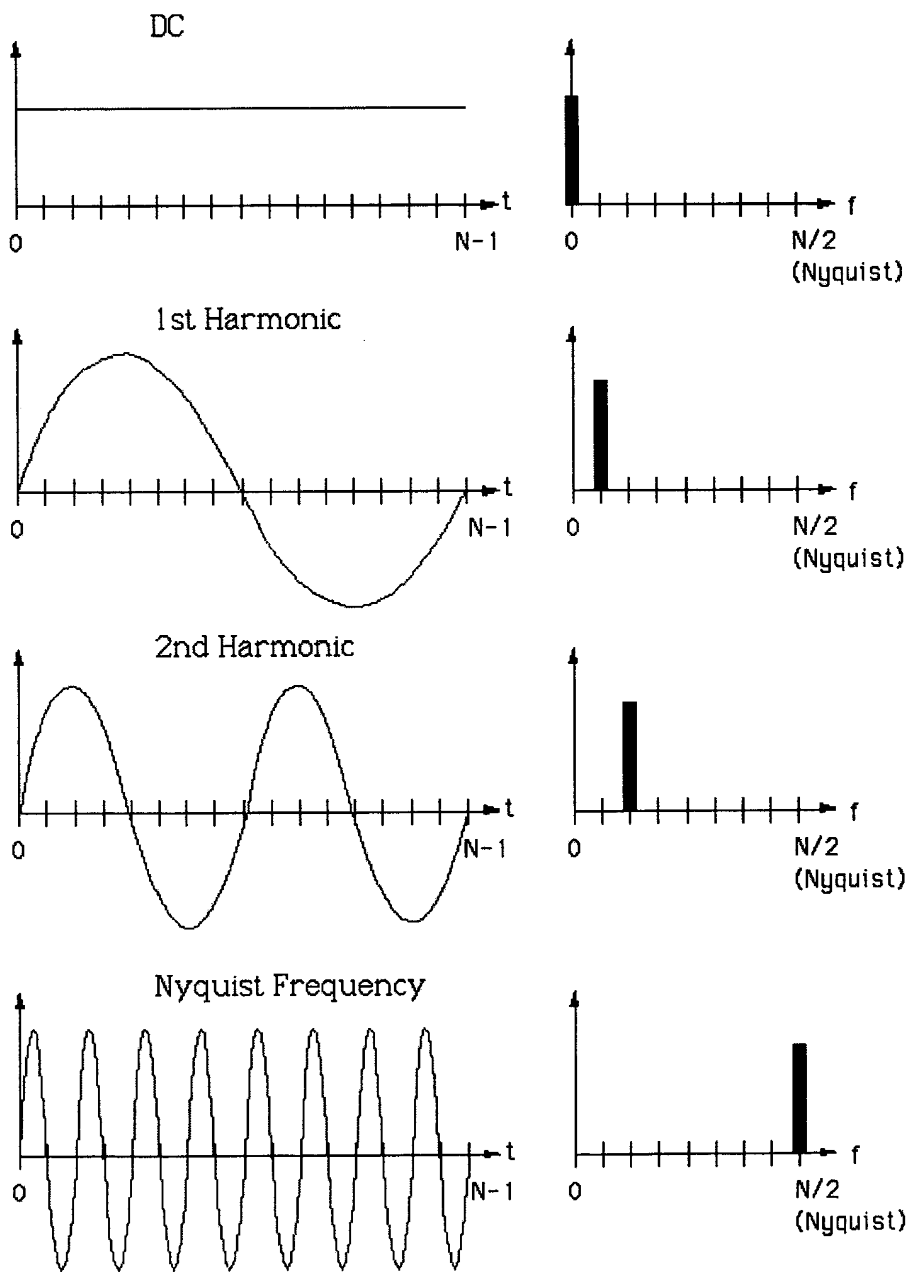

**Notes**

- The first sample X(0) of the transformed series is the DC component, more commonly known as the average of the input series.

- The DFT of a real series, ie: imaginary part of x(k) = 0, results in a symmetric series about the Nyquist frequency. The negative frequency samples are also the inverse of the positive frequency samples.

- The highest positive (or negative) frequency sample is called the Nyquist frequency. This is the highest frequency component that should exist in the input series for the DFT to yield "uncorrupted" results. More specifically if there are no frequencies above Nyquist the original signal can be **exactly** reconstructed from the samples.

- The relationship between the harmonics returns by the DFT and the periodic component in the time domain is illustrated below.




**DFT and FFT algorithm.**

While the DFT transform above can be applied to any complex valued series, in practice for large series it can take considerable time to compute, the time taken being proportional to the square of the number on points in the series. A much faster algorithm has been developed by Cooley and Tukey around 1965 called the FFT (Fast Fourier Transform). The only requirement of the the most popular implementation of this algorithm (Radix-2 Cooley-Tukey) is that the number of points in the series be a power of 2. The computing time for the radix-2 FFT is proportional to

$$N \log_2(N)$$

So for example a transform on 1024 points using the DFT takes 10 times longer than using the FFT, a significant speed increase. Note that in reality comparing speeds of various FFT routines is problematic, many of the reported timings have more to do with specific coding methods and their relationship to the hardware and operating system.

**Sample transform pairs and relationships**

- The Fourier transform is linear, that is

$$a\ f(t) + b\ g(t) \text{ ---> } a\ F(f) + b\ G(f)$$

$$a\ x_k + b\ y_k \text{ ---> } a\ X_k + b\ Y_k$$

- Scaling relationship

$$f(t / a) \text{ ---> } a\ F(a\ f)$$

$$f(a\ t) \text{ ---> } F(f / a) / a$$

- Shifting

$$f(t + a) \text{ ---> } F(f)\ e^{-j\ 2\ pi\ a\ f}$$

- Modulation

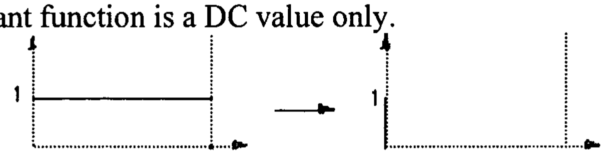
$$f(t)\ e^{j\ 2\ pi\ a\ t} \text{ ---> } F(t - a)$$

- Duality

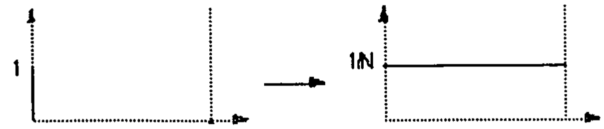
$$X_k \text{ ---> } (1/N)\ x_{N-k}$$

Applying the DFT twice results in a scaled, time reversed version of the original series.

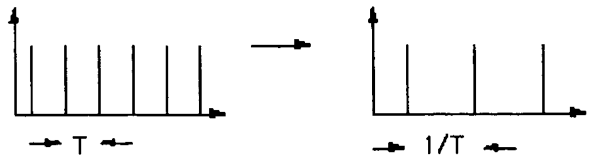
- The transform of a constant function is a DC value only.



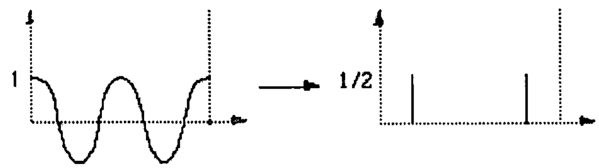

- The transform of a delta function is a constant



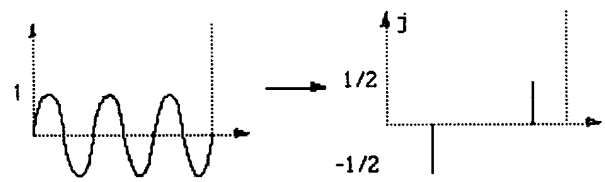

- The transform of an infinite train of delta functions spaced by T is an infinite train of delta functions spaced by 1/T.



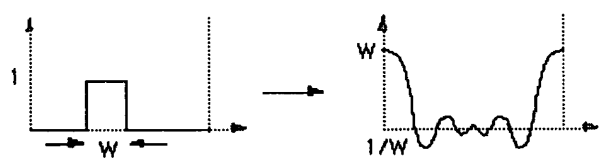

- The transform of a cos function is a positive delta at the appropriate positive and negative frequency.




- The transform of a sin function is a negative complex delta function at the appropriate positive frequency and a negative complex delta at the appropriate negative frequency.




- The transform of a square pulse is a sinc function




More precisely, if f(t) = 1 for |t| < 0.5, and f(t) = 0 otherwise then F(f) = sin(pi f) / (pi f)

- Convolution

$$f(t)\ \mathbf{x}\ g(t) \dashrightarrow F(f)\ G(f)$$

$$F(f)\ \mathbf{x}\ G(f) \dashrightarrow f(t)\ g(t)$$

$$x_k\ \mathbf{x}\ y_k \dashrightarrow N\ X_k\ Y_k$$

$$x_k\ y_k \dashrightarrow (1/N)\ X_k\ \mathbf{x}\ Y_k$$

- Multiplication in one domain is equivalent to convolution in the other domain and visa versa. For example the transform of a truncated sin function are two delta functions convolved with a sinc function, a truncated sin function is a sin function multiplied by a square pulse.

- The transform of a triangular pulse is a $\text{sinc}^2$ function. This can be derived from first principles but is more easily derived by describing the triangular pulse as the convolution of two square pulses and using the convolution-multiplication relationship of the Fourier Transform.

**Sampling theorem**

The sampling theorem (often called "Shannons Sampling Theorem") states that a continuous signal must be discretely sampled at least twice the frequency of the highest frequency in the signal.

More precisely, a continuous function f(t) is completely defined by samples every 1/fs (fs is the sample frequency) if the frequency spectrum F(f) is zero for f > fs/2. fs/2 is called the Nyquist frequency and places the limit on the minimum sampling frequency when digitising a continuous sugnal.

If x(k) are the samples of f(t) every 1/fs then f(t) can be **exactly** reconstructed from these samples, if the sampling theorem has been satisfied, by

$$f(t) = \sum_{k=-\infty}^{k=\infty} x(k)\ \text{sinc}(t\ fs - k)$$

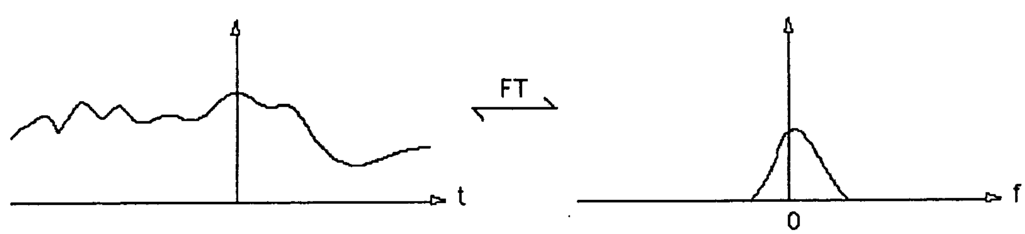
where

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

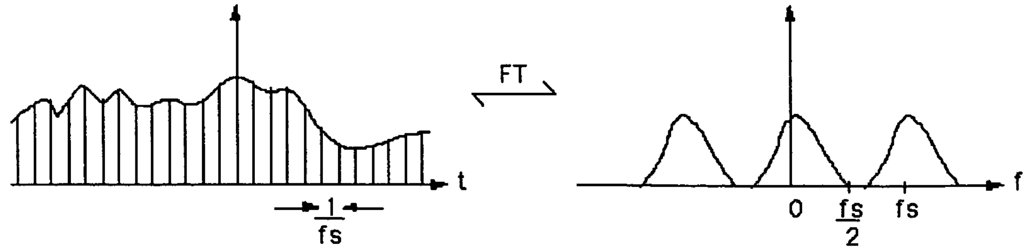
Normally the signal to be digitised would be appropriately filtered before sampling to remove higher frequency components. If the sampling frequency is not high enough the high frequency components will wrap around and appear in other locations in the discrete spectrum, thus corrupting it.

The key features and consequences of sampling a continuous signal can be shown graphically as follows.

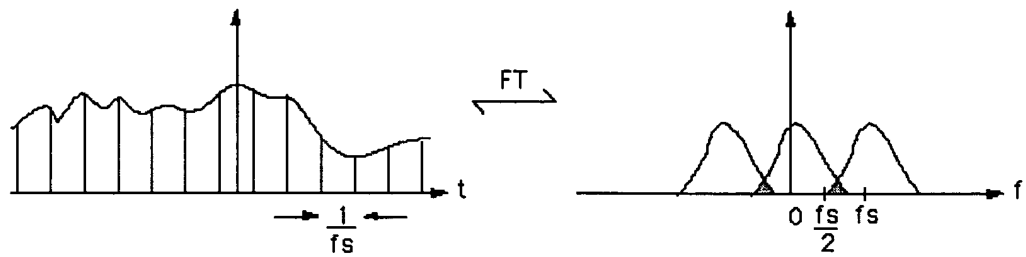
Consider a continuous signal in the time and frequency domain.




Sample this signal with a sampling frequency $f_s$, time between samples is $1/f_s$. This is equivalent to convolving in the frequency domain by delta function train with a spacing of $f_s$.




If the sampling frequency is too low the frequency spectrum overlaps, and become corrupted.




Another way to look at this is to consider a sine function sampled twice per period (Nyquist rate). There are other sinusoid functions of higher frequencies that would give exactly the same samples and thus can't be distinguished from the frequency of the original sinusoid.